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MARCINKIEWICZ INTEGRALS, HARMONIC MEASURE AND SOME REMOVABILITY PROBLEMS

IGNACIO URIARTE-TUERO

ABSTRACT. We deal with some applications of Marcinkiewicz integrals to problems related to harmonic measure on the one hand, and to removability problems for Sobolev spaces and quasiconformal mappings on the other hand. Techniques of this type have been used by Carleson, Jones, Makarov, Smirnov... to address these problems, and as a general program to understand harmonic functions on complicated domains in \mathbb{R}^n .

1. INTRODUCTION

We review some results on the relationship between harmonic measure, Marcinkiewicz integrals and Sobolev and quasiconformal removability problems, and announce some new results, with some comments on the proof.

The paper is structured as follows: Section 2 covers a preliminary background on harmonic measure for the benefit of the reader who is not familiar with the subject. Section 3 covers some specifics of the relationships between harmonic measure, Marcinkiewicz integrals and Sobolev and quasiconformal removability problems. In particular it reviews some of the fundamental theorems by Jones and Makarov, gives a proof of Beurling's estimate for the convenience of the reader, and announces (and briefly comments on the proof of) results that appeared in the author's thesis in this area.

Letters such as A, B, C, c_1 , etc. denote positive constants. The same letter in two sides of an inequality need not denote the same constant. The symbol $X \sim Y$ means that there are constants A, B such that $X \leq AY \leq BX$.

2. BACKGROUND ON HARMONIC MEASURE

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with sufficiently large boundary $\partial\Omega$ (e.g. assume that the Hausdorff dimension $\dim_H(\partial\Omega) > n - 2$). Then the Dirichlet problem can be solved on Ω , i.e., for any continuous $F \in C(\partial\Omega)$ there is a harmonic function \tilde{F} ($\Delta\tilde{F} = 0$) on Ω with $\tilde{F} = F$ on $\partial\Omega$. (Technical problems arise if $n > 2$ and Ω is unbounded). The Riesz representation theorem shows that there exists a unique probability measure ω_z (so called harmonic measure) supported on $\partial\Omega$ such that, for $F \in C(\partial\Omega)$ and $z \in \Omega$,

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$$(2.1) \quad \tilde{F}(z) = \int_{\partial\Omega} F(x) d\omega_z(x).$$

Forgetting technical problems, whenever this procedure works, the harmonic measure of $A \subseteq \partial\Omega$ with respect to $z \in \Omega$, denoted by $\omega(z, A, \Omega) = \omega_z = \omega$, can be defined. As a function of z , it is harmonic (hence the name), it satisfies $0 \leq \omega(z, A, \Omega) \leq 1$, and, by Harnack's inequality, for every $z_1, z_2 \in \Omega$, there exists a constant $c = c(z_1, z_2)$ such that

$$(2.2) \quad \frac{1}{c}\omega(z_1, A) \leq \omega(z_2, A) \leq c\omega(z_1, A).$$

In other words, if $z_1, z_2 \in \Omega$, then ω_{z_1} and ω_{z_2} are mutually absolutely continuous.

A physical interpretation is that $\omega(z, A, \Omega)$ is the Coulombic potential at $z \in \Omega$ when the distribution of charges in $\partial\Omega$ is given by $F = \chi_A$, the characteristic function of A . We will mainly address here the situation of very irregular surfaces (which arise, e.g. in batteries).

The harmonic measure ω_z has different equivalent descriptions depending on the situation. In the case of the disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we have that

$$(2.3) \quad \omega_0(A) = \frac{1}{2\pi} \int_A |d\zeta| = \frac{\Lambda(A)}{2\pi}$$

where $A \subset \mathbb{T} = \partial\mathbb{D}$, and $\Lambda(A)$ is the length (one dimensional Hausdorff measure of A), i.e. it is the normalized total angle under which the set A is seen from 0. Harmonic measure is conformally invariant, so, if G is a simply connected domain with locally connected boundary and f maps \mathbb{D} conformally onto G , then f extends continuously to the closed unit disk, $\overline{\mathbb{D}}$ by Caratheodory's theorem, and then, for a Borel set $A \subset \partial G$ we have that harmonic measure is given by the push forward of normalized Lebesgue measure on \mathbb{T} , i.e.

$$(2.4) \quad \omega(z, A, G) = \omega(f^{-1}(z), f^{-1}(A), \mathbb{D}).$$

In particular, if $G = \mathbb{D}$, then, taking as f a Moebius automorphism of \mathbb{D} , we get from (2.3) and (2.4) that harmonic measure is given by the Poisson integral of the characteristic function of A , i.e.

$$(2.5) \quad \omega(z, A, \mathbb{D}) = \frac{1}{2\pi} \int_A \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|.$$

We already mentioned above the definition of harmonic measure for “nice” domains in \mathbb{R}^n as it arises in the solution of the Dirichlet problem, as in (2.1). A consequence of that formula is that, if $F = \chi_A$ for $A \subset \partial\Omega$, then $\tilde{F}(z) = \omega_z(A)$.

If $\partial\Omega \subset \mathbb{R}^n$ is smooth, then harmonic measure is absolutely continuous with respect to surface measure $d\omega_z \ll d\sigma$, and its Radon-Nikodym derivative with respect to surface measure is given by the normal derivative of the Green's function of Ω with pole at z , i.e. $\frac{d\omega_z}{d\sigma} = \frac{-\partial G(\zeta, z)}{\partial n_\zeta}$.

For a compactly supported planar signed measure μ one can define its energy as

$$(2.6) \quad I(\mu) = \int \int \log \frac{1}{|z - \zeta|} d\mu(\zeta) d\mu(z),$$

provided that

$$(2.7) \quad \int \int \left| \log \frac{1}{|z - \zeta|} \right| d|\mu|(\zeta) d|\mu|(z) < \infty.$$

Then, if $P(E)$ is the set of all Borel probability measures supported on the compact set E , there exists a unique measure μ_E so that $I(\mu_E) = \inf \{I(\sigma) : \sigma \in P(E)\}$, called the equilibrium distribution of E . And if the compact set $E = \Omega^c$, then $\mu_E = \omega(\infty, \cdot, \Omega)$.

In the context of complex dynamics, harmonic measure also is important, since it is the invariant measure for a Julia set of a polynomial. A particularly intuitive description of harmonic measure was proved by Kakutani, namely that $\omega(z, A, \Omega)$ is the probability that a Brownian motion conditioned to start at z will first hit $\partial\Omega$ in A .

3. HARMONIC MEASURE, MARCINKIEWICZ INTEGRALS AND REMOVABILITY PROBLEMS

Many problems in potential theory and complex analysis can be reduced to estimating $\omega(z, A, \Omega)$ in terms of geometric information about the set A (such as the smoothness of $\partial\Omega$, or the Hausdorff dimension $\dim_H(A)$). In this direction, Øksendal [Ø1] showed that ω and m_n (Lebesgue n -measure) are always mutually singular ($\omega \perp m_n$). If $\partial\Omega$ is very complicated, most of it is “hidden” and hence unlikely to be hit first by Brownian motion, so Øksendal [Ø2] conjectured that in \mathbb{R}^2 , harmonic measure is singular with respect to the s -Hausdorff measure ($H^s \perp \omega$) for $s > 1$. Carleson [C1] proved it for Cantor sets with dynamical methods. Then, Makarov [M] proved it for simply connected domains $\Omega \subseteq \mathbb{R}^2$. He actually showed much more, since he gave a gauge function for the support of ω , showing that $\dim_H(\text{supp}(\omega)) = 1$. Jones and Wolff [JW] solved the problem for general domains in \mathbb{R}^2 by proving that $\dim_H(\text{supp}(\omega)) \leq 1$. Later Wolff [W1] showed that $\text{supp}(\omega)$ is contained in a set of σ -finite H^1 measure.

In the case of \mathbb{R}^n , Bourgain [B] showed that there is a constant $\omega(n) < n$ such that $\omega \perp H^s$ for $s > \omega(n)$. Hence, $\omega(2) = 1$. Wolff [W2] showed that $\omega(3) > 2$, and thus $\omega(n) > n - 1$ for any $n \geq 3$. Carleson, Jones and Makarov ([CJ], [JM]) studied ω using geometry and complex analysis. Carleson and Jones ([CJ]) used an estimate essentially due to Beurling (generalized by Jones and Makarov in [JM]) and suggested studying Beurling’s estimate by using the Marcinkiewicz integrals I_λ , where $\lambda > 0$, (see [S]) which were refined for $\lambda = 0$ by Jones and Makarov ([JM]) to get sharp estimates for ω . More precisely (see [JM]), Beurling (essentially) showed that if $\Omega \subset \mathbb{C}$ is a simply connected domain, and ω_a is the harmonic measure with basepoint $a \in \Omega$, assuming $|\zeta - a| \geq 1, r \leq \frac{1}{2}$, we have

$$(3.1) \quad \omega_a B(\zeta, \frac{r}{2}) \leq C \exp \left\{ -\frac{1}{2} \int_r^1 \frac{dt}{d(\zeta, t)} \right\}$$

where

$$d(t, \zeta) = \max\{\delta(z) : z \in \Omega, |\zeta - z| = t\},$$

and

$$\delta(z) = \text{dist}(z, \partial\Omega),$$

and C is an absolute positive constant. Geometrically, $d(t, \zeta)$ is comparable to the sidelength of the largest Whitney cube (in the Whitney decomposition of Ω) intersecting the set $\{z \in \Omega, |\zeta - z| = t\}$.

We will first sketch a relationship between Beurling's formula and the integrals of Marcinkiewicz, as developed in [JM], to get very sharp results on harmonic measure (and on many other related mathematical objects). Later we will give a sketch of the proof of Beurling's formula (essentially proved in [CJ]), in case the reader is not familiar with it. It should be noted that [JM] generalize Beurling's formula for the non-simply connected or multidimensional case and use its relation with Marcinkiewicz integrals in the way we will describe for the simply connected case to get the aforementioned results on harmonic measure also in these more general cases.

In \mathbb{R}^2 , (the definition is similar in \mathbb{R}^n), the Marcinkiewicz integral I_λ with $\lambda > 0$ is defined as

$$(3.2) \quad I_\lambda(\zeta) = \int_{\Omega} \frac{\delta^\lambda(z)}{|z - \zeta|^{2+\lambda}} dm_2(z),$$

These integrals satisfy L^1 and BMO-type estimates, namely (see [S],[Zy]), $I_\lambda(\zeta) < \infty$ almost everywhere on Ω^c , and

$$(3.3) \quad \int_{\Omega^c} I_\lambda(\zeta) dm_2(\zeta) \leq \frac{C}{\lambda} m_2(\Omega)$$

where m_2 is the planar Lebesgue measure, and if, say, we consider our universe to be $[0, 1]^2$, i.e. take $\Omega \subset [0, 1]^2$ and $\Omega^c \subset [0, 1]^2$, then, with universal constants,

$$(3.4) \quad m_2\{\zeta \cap [0, 1]^2 : I_\lambda(\zeta) > t\} \leq C m_2(\Omega) \exp\{-ct\lambda\}$$

By changing to polar coordinates and just keeping the contribution of the largest Whitney cube hit at each radius, one can see that, for an absolute constant c_1 ,

$$(3.5) \quad I_\lambda(\zeta) \geq c_1 \int_0^\infty \frac{d^{1+\lambda}(\zeta, t) dt}{t^{2+\lambda}}.$$

Now, as in [JM], using first Hölder's inequality and then (3.5), we get that

$$(3.6) \quad \begin{aligned} |\log r| = \int_r^1 \frac{dt}{t} &\leq \left(\int_r^1 \frac{d^{1+\lambda}(\zeta, t) dt}{t^{2+\lambda}} \right)^{\frac{1}{2+\lambda}} \left(\int_r^1 \frac{dt}{d(\zeta, t)} \right)^{\frac{1+\lambda}{2+\lambda}} \leq \\ &\leq (c_1^{-1} I_\lambda(\zeta))^{\frac{1}{2+\lambda}} \left(\int_r^1 \frac{dt}{d(\zeta, t)} \right)^{\frac{1+\lambda}{2+\lambda}} \end{aligned}$$

Therefore, using (3.1), [JM] get

$$(3.7) \quad \omega_a B(\zeta, \frac{r}{2}) \leq C \exp \left\{ -\text{const.} [I_\lambda(\zeta)]^{-\frac{1}{1+\lambda}} |\log r|^{\frac{2+\lambda}{1+\lambda}} \right\}$$

From here, it is clear that upper bounds for the Marcinkiewicz integrals I_λ (as in (3.3) and (3.4)) will give upper bounds for harmonic measure. The best estimates using this approach would be obtained if one could make $\lambda = 0$, however the most obvious attempt to do so, namely to substitute $\lambda = 0$ on (3.2) (let us denote such an integral by I_0^0) is hopeless, since one can build a Cantor type set $K = \Omega^c \subset [0, 1]^2$ so that $I_0^0(\zeta) = \infty$ on all ζ . A further attempt, namely to substitute $\lambda = 0$ on the right hand side of (3.5) (making the upper limit of integration equal to 1 instead of ∞ if $\Omega^c \subset [0, 1]^2$, since that is the most relevant region of integration), denoted by I_0 , is also hopeless, since one can build a set $\Omega \subset [0, 1]^2$ consisting of a union of vertical strips, so that $\int_{\Omega^c} I_0 = \infty$.

A “correct” definition for the case $\lambda = 0$ seems to be ([JM])

$$(3.8) \quad \tilde{I}_0(\zeta) = \inf_{\Delta} \int_{\Delta} \frac{d(t, \zeta)}{t^2} dt,$$

where the inf is taken over all $\Delta \subset \mathbb{R}^+$ with logarithmic density at least $\frac{1}{2}$ in the sense that

$$(3.9) \quad \int_{\Delta \cap [r, 1]} \frac{dt}{t} \geq \frac{1}{2} \log \left(\frac{1}{r} \right) \text{ if } r \leq r_0.$$

By “correct” we mean that the analogues of (3.3) and (3.4) continue to hold. More explicitly, we have the following

Theorem (Jones, Makarov). *If $\Omega \subset [0, 1]^n$, then*

$$\begin{aligned} & \| \tilde{I}_0 \|_{L^1(\Omega^c \cap [0, 1]^n)} \leq C | \Omega | \\ & | \{ z \in \Omega^c \cap [0, 1]^n : \tilde{I}_0(z) > t \} | \leq C | \Omega | e^{-ct} \end{aligned}$$

With this theorem, and using the approach sketched above (as in (3.6), but the integrals instead of being on $[r, 1]$ are on $[r, 1] \cap \Delta$, and using (3.9)), one gets

Theorem (Jones, Makarov). *1) Let ω be a harmonic measure in \mathbb{R}^n . Then, m_n -a.e. the following holds:*

$$\forall M > 0, \quad \omega B(., r) = O \left(\exp \left\{ -M \log^{\frac{n}{n-1}} \frac{1}{r} \right\} \right), \text{ as } r \rightarrow 0.$$

2) For any function $M = M(r)$ satisfying $M(r) \rightarrow \infty$ as $r \rightarrow 0$, there exists a domain $\Omega \subset \mathbb{R}^n$ with $m_n(\partial\Omega) > 0$, and for any $x \in \partial\Omega$,

$$\omega B(x, r) \geq \exp \left\{ -M(r) \log^{\frac{n}{n-1}} \frac{1}{r} \right\},$$

as $r \rightarrow 0$, infinitely often.

Theorem (Jones, Makarov). *Let $\chi = \chi(t)$ be an increasing function in a neighborhood of $+\infty$. Then*

$$\omega B(x, r) \leq \exp \left\{ -\chi \left(\log \log \frac{1}{r} \right) \left(\log \frac{1}{r} \right)^{\frac{n}{n-1}} \right\},$$

infinitely often, m_n -a.e., for every harmonic measure ω in \mathbb{R}^n , if and only if

$$\int_{\chi(t)^{n-1}}^{\infty} \frac{dt}{t} = \infty$$

Theorem (Jones, Makarov). 1) Let ω be the harmonic measure of any domain in \mathbb{R}^n . Then, for any $K \geq 1$, the relation

$$\omega B(., r) = O(r^K)$$

holds everywhere except for a set of dimension $n - c_1 K^{-n+1}$, where $c_1 = c_1(n)$ is a positive dimensional constant.

2) For any $K \geq n$, there exists a domain $\Omega \subset \mathbb{R}^n$ satisfying

$$\dim \partial \Omega \geq n - c_2(n) K^{-n+1},$$

and

$$\omega B(x, r) \geq \text{const. } r^K, \text{ for all } x \in \partial \Omega.$$

Using these powerful tools, Jones and Makarov were also able to get applications to questions about conformal mapping, more particularly questions about singularity of boundary distortion, area and dimension of the boundary, and integral means (see [JM]).

Remark. Let us now sketch a proof of Beurling's formula, in case the reader is not familiar with it. The formula is a consequence of a theorem due to Beurling, however, the argument we will present appears (essentially) in [CJ]. Somewhat independently, while discussing about Beurling's formula and its applications in [JM], Garnett and Koosis (personal communications) showed us essentially the same proof, but with some variations that we will also sketch.

Proof. (Of Beurling's formula in the simply connected case). For a simply connected $\Omega \subset \mathbb{C}$, pick $a \in \Omega$, let $\zeta \in \Omega^c$, with $\text{dist}(a, \zeta) \geq 1$, and let $\tau = \{z \in \Omega : |z - \zeta| = r_0\}$, with r_0 small in comparison to $\text{dist}(a, \zeta)$. For each r , $r_0 < r < \text{dist}(a, \zeta)$, let γ_r be the arc about ζ of radius r , lying in Ω (save for its endpoints), separating a from ζ . Then $d(\zeta, r) \geq \max\{\text{dist}(z, \partial \Omega) : z \in \gamma_r\}$. Then, for any $z_0 \in \tau$, denoting by G_Ω the Green's function in Ω , we have

$$(3.10) \quad G_\Omega(z_0, a) \leq C \exp \left\{ -C_0 \int_{r_0}^{\text{dist}(a, \zeta)} \frac{dr}{d(\zeta, r)} \right\}$$

To see (3.10) we will give the following argument shown by Koosis (which is the same argument that had appeared before in [CJ], see also [GM] pp.120ff). Map Ω conformally onto \mathbb{D} , taking a to 0 and z_0 to $b \in [0, 1]$. Then $G_\Omega(z_0, a) = G_\mathbb{D}(0, b)$, and $h_\Omega(z_0, a) = h_\mathbb{D}(0, b)$, the hyperbolic distance from 0 to b in \mathbb{D} .

Then $G_\mathbb{D}(0, b) = \log\left(\frac{1}{b}\right) \sim 1 - b$, as long as $|b|$ is not small. Also, as long as $|b|$ is not small,

$$h_\mathbb{D}(0, b) = \int_0^b \frac{dx}{1-x^2} = \frac{1}{2} \log \frac{1+b}{1-b} = \log \frac{1}{\sqrt{1-b}} + O(1)$$

Hence, $G_\Omega(z_0, a) = G_\mathbb{D}(0, b) \sim e^{-2h_\mathbb{D}(0, b)} = e^{-2h_\Omega(z_0, a)}$.

In order to find $h_\Omega(z_0, a)$, integrate the element of hyperbolic length in Ω along a hyperbolic geodesic Γ in Ω , joining a to z_0 . Notice that the element of hyperbolic length at any point $z \in \Omega$ is $|f'(z)||dz|$, where $f : \Omega \rightarrow \mathbb{D}$ is the Riemann map

taking z to 0. From Schwarz' lemma and Koebe's $\frac{1}{4}$ theorem, we know that, for two (known) numerical constants $A(=\frac{1}{4})$ and B ,

$$\frac{A}{\text{dist}(z, \partial\Omega)} \leq |f'(z)| \leq \frac{B}{\text{dist}(z, \partial\Omega)}$$

Using polar coordinates with pole at ζ , we have $|dz| \geq dr$ along Γ , and if Γ crosses γ_r at z_r , then $\text{dist}(z_r, \partial\Omega) \leq d(\zeta, r)$, this being best possible. So $h_\Omega(z_0, a) \geq A \int_{r_0}^{\text{dist}(z_0, a)} \frac{dr}{d(\zeta, r)}$, yielding (3.10). (Notice that, if Ω is symmetric with respect to the line joining z_0 and a , then Γ can be taken to be the line segment joining those two points, and we have the reverse inequality, with B standing in place of A , i.e. (3.10) is sharp).

The other important element in the proof of Beurling's formula that we are sketching is the relation

$$(3.11) \quad \omega_a B(\zeta, \frac{r}{2}) \leq C \max\{G_\Omega(z_1, a) : z_1 \in \Omega, |\zeta - z_1| = r\}$$

The proof of (3.11) can be given essentially by Pfluger's estimate ([CJ],[JM]), however we will present an argument shown to us by Garnett (see also [GM] pp.120ff), and comment on a variation of it due to Koosis. Although Beurling's formula as stated is valid for simply connected domains, the relation (3.11) holds in a more general setting (namely Ω a domain with $\text{cap}(\Omega^c) > 0$), and we will prove it in such a setting.

As a first step, consider Ω a bounded finitely connected Jordan domain with smooth boundary ($C^{1+\alpha}$ with $\alpha > 0$ will do, see e.g. [GM].) Let $\psi \in C^\infty(B(\zeta, \frac{3r}{4}))$, $0 \leq \psi \leq 1$, $\psi(z) = 1$ on $B(\zeta, \frac{r}{2})$, and $|\Delta\psi| \leq \frac{C}{r^2}$ (which can be obtained by rescaling a bump function supported on $B(\zeta, 1)$.)

Now, since $\partial\Omega$ is $C^{1+\alpha}$, and since $\frac{\partial G_\Omega(a, \xi)}{\partial n_\xi} < 0$ we get

$$\begin{aligned} \omega_a(\partial\Omega \cap B(\zeta, \frac{r}{2}), \Omega) &= \int_{\partial\Omega \cap B(\zeta, \frac{r}{2})} -\frac{\partial G_\Omega(a, \xi)}{\partial n_\xi} \frac{ds(\xi)}{2\pi} \leq \\ &\leq -\frac{1}{2\pi} \int_{\partial\Omega} \psi(\xi) \frac{\partial G_\Omega(a, \xi)}{\partial n_\xi} ds(\xi) = \frac{1}{2\pi} \int \int_\Omega G_\Omega(a, \xi) \Delta\psi(\xi) d\xi \end{aligned}$$

by Green's formula, taking into account the vanishing properties of ψ and the harmonicity and boundary values of $G_\Omega(a, \xi)$.

Now, since $\psi = 0$ outside $B(\zeta, \frac{3r}{4})$, and since $G_\Omega(a, \xi)$ is harmonic in $\xi \in B(\zeta, \frac{3r}{4}) \cap \Omega$ and extends continuously to $\overline{B(\zeta, \frac{3r}{4})} \cap \overline{\Omega}$, and $G_\Omega(a, \xi) = 0$ if $\xi \in \partial\Omega$, by the maximum principle we get

$$\begin{aligned} \frac{1}{2\pi} \int \int_\Omega G_\Omega(a, \xi) \Delta\psi(\xi) d\xi &= \frac{1}{2\pi} \int \int_{\Omega \cap B(\zeta, \frac{3r}{4})} G_\Omega(a, \xi) \Delta\psi(\xi) d\xi \leq \\ &\leq \frac{1}{2\pi} \frac{C}{r^2} \pi \left(\frac{3r}{4}\right)^2 \max_{\xi \in \overline{B(\zeta, \frac{3r}{4})} \cap \overline{\Omega}} G_\Omega(a, \xi) = C \sup_{\substack{|\zeta - \xi| = \frac{3r}{4} \\ \xi \in \Omega}} G_\Omega(a, \xi). \end{aligned}$$

which finishes the proof in case Ω is a bounded finitely connected domain with $C^{1+\alpha}$ boundary.

Now we have to approximate a general Ω (meaning $\text{cap}(\Omega^c) > 0$) by such domains. Since the estimates and constants are independent of the domain, the result

will follow from the case we just proved. As a remark, if Ω were a simply connected planar domain, we could easily approximate Ω by $\Omega_r = f(\mathbb{D}_r)$ where $f : \mathbb{D} \rightarrow \Omega$ is the Riemann map and $\mathbb{D}_r = \{|z| < r\}$ for $r < 1$.

In the general case, if C_n denotes the collection of closed dyadic cubes of size 2^{-n} in the plane, we can take $\mathcal{D}_n = \bigcup_{Q \in C_n, Q \subset \Omega} Q$. Then \mathcal{D}_n are finitely connected Jordan domains which increase to Ω . There is a subsequence of them so that $\partial\mathcal{D}_{n_j} \cap \partial\mathcal{D}_{n_{j+1}} = \emptyset$. Take that subsequence and fit a C^2 curve in between $\partial\mathcal{D}_{n_j}$ and $\partial\mathcal{D}_{n_{j+1}}$ (e.g. by using splines or by taking the surface measure on $\partial\mathcal{D}_{n_{j+1}}$, considering F_1 its Riesz potential of order 1 and applying the implicit function theorem when sufficiently close to $\partial\mathcal{D}_{n_{j+1}}$, in order to avoid the critical points of F_1 , which gives a C^∞ curve.) In any case, we have a sequence of bounded finitely connected Jordan domains with smooth boundary $\Omega_n \subset \overline{\Omega_n} \subset \Omega_{n+1}$ which increases to Ω (with the obvious modification if Ω is not bounded.)

Then, by Harnack and the maximum principle, $G_{\Omega_n}(z, a) \nearrow G_\Omega(z, a)$, assuming $\text{cap}(\Omega^c) > 0$. Also, $\omega_{\Omega_n}(a, \cdot) \rightharpoonup \omega_\Omega(a, \cdot)$, i.e. we have weak-star convergence, and the weak-star limit is unique and does not depend on the sequence Ω_n . Consequently, since the constants do not depend on the domain, and weak-star convergence of measures is lower semicontinuous on open sets, we get

$$\omega_a(\partial\Omega \cap B(\zeta, \frac{r}{2}), \Omega_n) \leq C \sup_{\substack{|\zeta - \xi| = \frac{3r}{4} \\ \xi \in \Omega_n}} G_{\Omega_n}(a, \xi) \leq C \sup_{\substack{|\zeta - \xi| = \frac{3r}{4} \\ \xi \in \Omega}} G_\Omega(a, \xi)$$

and passing through an intermediate open ball with center ζ and radius $\frac{5r}{8}$ to make use of the aforementioned lower semicontinuity, we get the desired conclusion (3.11).

P. Koosis showed us another proof which is a variation of this argument, which we sketch now. Take the same bump function ψ . Apply Jensen's formula (consider a smooth function as a difference of subharmonic functions thinking of their Laplacian) to get

$$\psi(a) = \int_{\partial\Omega} \psi(\xi) d\omega_{a,\Omega}(\xi) - \frac{1}{2\pi} \int \int_{\Omega} (\Delta\psi)(\xi) G_\Omega(\xi, a) dx dy$$

Now use that $\psi \geq 0$, that $\psi = 0$ for $\xi \in \partial\Omega$ with $|\xi - \zeta| > r$ and $\psi = 1$ for $\xi \in \partial\Omega$ with $|\xi - \zeta| < \frac{r}{2}$ to get

$$\omega_{a,\Omega}(B(\zeta, \frac{r}{2})) \leq \frac{1}{2\pi} \int \int_{\Omega} (\Delta\psi)(\xi) G_\Omega(\xi, a) dx dy$$

and again estimate the integral over Ω by the integral over $B(\zeta, \frac{3r}{2})$, and proceed as before. \square

Jones and Smirnov, [JS], used tools related to \tilde{I}_0 to get sufficient conditions for the Sobolev and (quasi)conformal removability problems. These kind of problems and their applications and tools had been studied, among others, in [AB, Be, Bi, C1, C2, G, GS1, GS2, Je, Jo, Ka, Ko, KW, PR, U, W]. Please see also the references for a necessarily incomplete list of other related papers, and [AH], [IM], [Ma] as general references for ideas used in these problems.

Let us define quasiconformal and conformal removability in our context. For a domain $U \subseteq \mathbf{R}^n$, a compact set $K \subset U$ is (quasi)conformally removable inside U if any f homeomorphism of U and (quasi)conformal on $U \setminus K$ is actually (quasi)conformal on U . Similarly, $K \subset U \subseteq \mathbf{R}^n$ is Sobolev $W^{1,n}$ -removable if any

$f \in C(U) \cap W^{1,n}(U \setminus K)$ is actually in $W^{1,n}(U)$. (The problems are quite different from those when f is not assumed to be globally continuous. But it is the formulation with global continuity the one that appears naturally in complex dynamics and has applications in that area related to dynamical removability.) Roughly speaking, a removable set K is sufficiently regular that a continuous extension implies a Sobolev or (quasi)conformal extension (but it is not just a question of size, since one can get removable sets of dimension 2 in the plane). Both definitions of removability (quasiconformal, Sobolev) do not depend on the open set U . In \mathbf{R}^2 , solving Beltrami (see [Jo]) gives K is quasiconformally removable if and only if it is conformally removable. Sobolev removability implies quasiconformal removability. In both cases, if K is removable then $|K| = 0$ (area zero). (See e.g. [A, IM, U, Zi] for background on Sobolev and quasiconformal mappings).

As we mentioned above, Jones and Smirnov ([JS]), using \tilde{I}_0 proved that, if $\Omega \subset \mathbb{C}$ is simply connected and the Riemann map $\phi : \mathbb{D} \rightarrow \Omega$ has modulus of continuity

$$|\phi(x) - \phi(y)| \leq \omega_\phi(|x - y|)$$

then

$$\begin{aligned}
 (3.12) \quad \omega_\phi(t) &< \exp\left(-\sqrt{\log \frac{1}{t} \log \log \frac{1}{t}/o(1)}\right) \text{ as } t \rightarrow 0 \Rightarrow \\
 &\Rightarrow K = \partial\Omega \text{ is conformally and } W^{1,2}\text{-removable}
 \end{aligned}$$

By [JM] we know that

$$\int_0 \left| \frac{\log \omega_\phi(t)}{\log t} \right|^2 \frac{dt}{t} = \infty \Leftrightarrow |\partial\Omega| = |K| = 0$$

which in particular, stopping at the $\log \log$ term, implies that

$$(3.13) \quad \omega_\phi(t) < \exp\left(-\sqrt{\log \frac{1}{t} / \log \log \frac{1}{t}}\right) \Rightarrow |K| = 0$$

However, for any $\varepsilon > 0$, there is an Ω , such that

$$\begin{aligned}
 (3.14) \quad \omega_\phi(t) &< \exp\left(-\sqrt{\log \frac{1}{t} / \left(\log \log \frac{1}{t}\right)^{1+\varepsilon}}\right) \Rightarrow \\
 &\Rightarrow |\partial\Omega| > 0 \Rightarrow \partial\Omega \text{ is non-removable.}
 \end{aligned}$$

So the conjecturally sharp condition (at least stopping at the $\log \log$ term) for quasiconformal and Sobolev removability is the one in (3.13). It is also conjectured that if $\partial\Omega$ has zero area sets then it is removable, but that is a much harder question. It should be mentioned that P. Koskela and T. Nieminen ([KN]) have been able to remove the $\log \log$ term in (3.12) but their method does not allow to put that term downstairs as in (3.13). The method in [KN] is related to porosity, and is only on the surface different from that of [JS], but the ingredients, tools and geometry are actually the same in both methods.

Because of all these reasons, it is of interest to try to refine the arguments of [JS] or [KN]. In this direction, and with the aim of counting more precisely the Whitney cubes that appear in the argument of [JS], the author has considered, in his Ph.D. thesis ([UT]) \tilde{I}_0 with a density parameter α replacing $\frac{1}{2}$, so the inf in the definition of \tilde{I}_0 (see (3.8) and (3.9)) is taken over sets Δ such that $\int_{\Delta \cap [r,1]} dt/t \geq (1-\alpha) \log(1/r)$ if $r \leq r_0$. (α is typically small, so the integral is over a set of big density). Also, the author considers instead of the Euclidean norm in \mathbb{R}^n , the sup norm (which fits

better the arguments of dyadic cubes, but does not change at all the applications of the theorem). Under those circumstances, he obtained the following

Theorem (U.T.). *If $\Omega \subset [0, 1]^n$, then if the definition of \tilde{I}_0 is taken over sets Δ such that $\int_{\Delta \cap [r, 1]} dt/t \geq (1 - \alpha) \log(1/r)$ if $r \leq r_0$, with $0 < \alpha < 1$, and considering the sup norm in \mathbb{R}^n ,*

$$(3.15) \quad \|\tilde{I}_0\|_{L^1(\Omega^c)} \leq \frac{C}{\alpha} \log\left(\frac{1}{\alpha}\right) |\Omega|$$

and the corresponding BMO estimate

$$(3.16) \quad |\{z \in \Omega^c \cap [0, 1]^n : \tilde{I}_0(z) > t\}| \leq C |\Omega| e^{-ct \frac{\alpha}{\log(\frac{1}{\alpha})}}$$

The techniques used in its proof are different from the ones in [JM], since [JM] count squares in annuli through a complicated argument, whereas [UT] uses a stopping time argument, which was to be expected to work given the BMO estimate. He does the dyadic case first and then applies the technique of averaging over all dyadic grids, (see [GJ]). This proof naturally breaks down the operator into bite-sized chunks, and provides an elementary view that shows exactly where each cube contributes to the integral. It is a more flexible proof than the previous proof in [JM] and it allows to change the density without further changes in the argument. It is important to consider a parameter α instead of using $\alpha = \frac{1}{2}$, as in [JM] since this provides a better understanding of the geometry and combinatorics involved, as mentioned before. Also, the geometry and combinatorics of the problem, that are more apparent in this proof, appear to have connections to the ones appearing in the Sobolev and quasiconformal removability problems, and also from this theorem one can recover the harmonic measure estimates in [JM] (as is done in [JM].) The full details of the proof are in [UT] and will appear elsewhere.

It is conjectured that the worst growth of $\|\tilde{I}_0\|_{L^1(\Omega^c)}$ in (3.15) is $\frac{C}{\alpha} |\Omega|$, which is what appears in a certain example, and this would give α instead of $\frac{\alpha}{\log(\frac{1}{\alpha})}$ in (3.16).

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